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Enlarged Lorentz–Dirac equations

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Abstract
Dirac's approach to incorporate the radiation into the equation of motion for a point charge in classical electrodynamics is based on three structural components: the point model for the electron, the Maxwell equations and the principle of relativity. These fundamental components lead to an equation of motion that involves an undetermined 4-vector B_μ. The Lorentz–Dirac equation corresponds to the case in which $B_\mu = 0$, but in general there is a large family of 4-vectors B_μ consistent with the above three basic components. This paper deals with the study of these equations of motion in the case of the three simplest permissible choices for B_μ. We show that these equations admit as exact solutions the motion of an arbitrary number of identical charges that are equally spaced in a circumference and that rotate at constant angular velocity. These solutions show that the rate of radiation emitted by the system of charges is completely independent of the 4-vector B_μ. We also study the restrictions over the dimensionless parameters that appear in the four-vectors B_μ, in order that the trajectories of the corresponding equations cannot be discriminated from the trajectory determined by the Lorentz–Dirac equation in a practical case, as for instance the design and operation of a synchrotron.

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1. Introduction

Dirac's approach to the equation of motion for the electron in classical electrodynamics is based on three essential components [1]. In the first one, the electron is modelled as a point charge from the very beginning. In the second one, Dirac assumes the validity of the Maxwell equations even at arbitrarily small distances from the electron. Finally, the third essential component is the principle of relativistic covariance, specially in what is concerned with the conservation of energy and momentum. Dirac arrived at the following equation:

$$\dot{v}^\mu = (e/mc)F^{\mu\lambda}v_\lambda + \tau_0\{\ddot{v}^\mu - \dot{v}^2 v^\mu/c^2\} + \dot{B}^\mu, \tag{1.1}$$

where e and m are the electron charge and mass, respectively, c is the speed of light, $F_{\mu\nu}$ is the external field acting on the electron and v_μ is its 4-velocity; the dots over the symbol denote proper time derivatives of an order equal to the number of dots, $\dot{v}^2 = \dot{v}^\alpha \dot{v}_\alpha$, the signature of the metric is +2, the system of units is the Gaussian one and τ_0 is given by

$$\tau_0 = \frac{2e^2}{3mc^3}, \tag{1.2}$$

which is a very small time, of the order of 10^{-23} s for an electron.

The 4-vector B_μ in equation (1.1) must be such that

$$v^\mu \dot{B}_\mu = 0. \tag{1.3}$$

In his paper Dirac makes the choice $B_\mu = 0$ based on a criterion of simplicity, the case in which equation (1.1) becomes the well-known Lorentz–Dirac equation. But it is important to emphasize that the hypothesis of simplicity is independent of the above-mentioned three essential components which by themselves are unable to determine a unique B_μ. For instance, Dirac exhibits in his paper a permissible B_μ that is proportional to the fourth power of the parameter τ_0 of equation (1.2).

The study of equation (1.1) with a non-zero 4-vector B_μ, named enlarged Lorentz–Dirac equations in this paper, has received little attention in the literature. It seems that the only exception is a paper by Eliezer [2], where this author remarks that a fully relativistic treatment must also consider the angular momentum conservation, in addition to the energy and linear momentum conservation taken into account in Dirac's paper. This requirement imposes the following additional restriction on the 4-vector B_μ:

$$v_\mu B_\lambda - v_\lambda B_\mu = \frac{\mathrm{d}A_{\mu\lambda}}{\mathrm{d}\tau}, \tag{1.4}$$

where $A_{\mu\lambda}(\tau)$ depends on the 4-velocity v_μ and its derivatives evaluated at the proper time τ. As Eliezer pointed out, the 4-vector B_μ proposed by Dirac does not fulfil equation (1.4), and consequently, it has to be discarded. In his paper Eliezer constructed B_μ proportional to τ_0^2, namely,

$$B_\mu(2) = k_2 \tau_0^2 \{\dot{v}^2 v_\mu / c^2 - (2/3)\ddot{v}_\mu\}, \tag{1.5}$$

where k_2 is an arbitrary dimensionless parameter. The 4-vector (1.5) is a permissible one since it satisfies equations (1.3) and (1.4) with

$$A_{\mu\lambda} = -(2/3)k_2\tau_0^2\{v_\mu \dot{v}_\lambda - v_\lambda \dot{v}_\mu\}. \tag{1.6}$$

The present paper deals with two problems associated with the enlarged Lorentz–Dirac equations. The first one consists in the construction of the 4-vector B_μ that follows in complexity to the one given by Eliezer in equation (1.5). We show that there is no B_μ proportional to τ_0^3, and we construct the two possible 4-vectors B_μ proportional to τ_0^4. The second problem consists in the study of exact solutions of the enlarged Lorentz–Dirac equations of Eliezer as well as the enlarged equations proposed here. It turns out that these equations, as the Lorentz–Dirac equation [3], admit as exact solutions the motion of an arbitrary number of identical charges that are equally spaced over a circumference and that rotate at constant angular velocity.

The existence of these exact analytic solutions is a remarkable mathematical result since to find exact solutions of highly nonlinear equations of motion is rather infrequent. The above-mentioned motion requires, of course, the assistance of appropriate external fields; these consist of a uniform time-independent magnetic field orthogonal to the plane where the charges are rotating, together with an electric field tangent to the circumference of motion and with a magnitude that takes a constant value on it. It is this relatively simple structure of the external fields which allows us to find the exact solutions presented in section 4. In turn, the external fields are generated by idealized sources. Thus, the electric field is generated by an infinitely long solenoid whose axis coincides with the z-axis and is fed with a current that increases linearly with time. Of course, this is not a practical solenoid, but practical aspects play here only a tangential role since our main interest consists in the study of the mathematical properties of the enlarged Lorentz–Dirac equations. Even if inside the solenoid there is a magnetic field that increases linearly with time, outside the solenoid, where the motion of the charges takes place, the magnetic field due to the solenoid is identically zero. Therefore, the magnetic field that holds the charges in a circular orbit must be generated by a source completely independent of the solenoid.

The exact solutions constitute a powerful tool to study the properties of the enlarged Lorentz–Dirac equations. For example, they allow us to obtain a clear and definite answer for the rate of radiation emitted by the system of charges, which can then be compared with the one obtained, independently of the equations of motion, by computing the flux of energy across the surface of a sphere that encloses the charges. It turns out that, as in the case of the Lorentz–Dirac equation, both results coincide, showing the full consistency of the enlarged Lorentz–Dirac equations with the Maxwell equations, as expected. The exact solutions also allow us to study the departure between the trajectories that the different equations of motion determine for an electron in the same external field. To this respect, the conclusion is that for the highest energies attainable for an electron in synchrotrons nowadays (which are less than 100 GeV) and for reasonable values of the dimensionless parameters that appear in the enlarged Lorentz–Dirac equations, the different trajectories cannot be discriminated by experimental means. In other words, even if from a strict mathematical point of view the trajectories are not the same, the differences between them are too small to play a role in the design and operation of a practical device such as a synchrotron.

A widely existing conception in classical electrodynamics is that under the influence of a given external electromagnetic field, an electron must have a perfectly well-defined and unique trajectory. This contrasts, however, with the actual situation, where several equations of motion have been proposed, which give rise to different trajectories in the same external field.

The equations of motion for a point charge can be classified into two categories. The first one consists in the equations of motion which, like the Lorentz–Dirac equation and the extended Lorentz–Dirac equations, are derived starting from the Maxwell equations as a fundamental component. The second category consists of the alternative equations of motion, like the equations proposed by Eliezer [4], Mo and Papas [5], Herrera [6], Landau and Lifshitz [7] and Bonnor [8]. None of these equations can be considered fully satisfactory. On one hand, the equations of the first category suffer of pathologies such as self-acceleration, where the electron is accelerated even in the absence of an electromagnetic field; these unphysical solutions are eliminated by imposing the vanishing of the acceleration in the remote future, but then the equations are in conflict with the principle of causality because the electron begins to accelerate before the force is actually applied. On the other hand, the equations in the second category, which have been proposed precisely to avoid the troubles of the equations of the Lorentz–Dirac equation type, are in conflict with the Maxwell equations because they lead to an incorrect rate of radiation [9–12]. This trouble of the alternative equations of motion seems to be worse than the acausalities that affect the equations of the Lorentz–Dirac type, which are of the order of 10^{-23} s and that are therefore too small to have any measurable consequence. According to this view, the use of the Maxwell equations is an imperative to obtain the equation of motion for a point electron, which then singularizes Dirac's approach.

In spite of that, Dirac himself remarks that his procedure does not lead to a unique equation of motion; this feature has been almost completely overlooked in the literature. The three essential components of Dirac's approach make Eliezer's equation (defined by (1.1) and (1.5)) as well as the equations in section 3 of this paper as permissible as the Lorentz–Dirac one. It is perhaps the preconception that there must be a 'correct equation of motion for a point charge' which has played a determinant role to this respect. The hypothesis of simplicity of Dirac appears to be direct, simple and natural, but certainly it does not have the same physical status and strength as the three essential components, being rather a choice or prescription.

There are several derivations of the Lorentz–Dirac equations in the literature that do not use Dirac's hypothesis of simplicity; however, a close examination of these derivations shows that they contain assumptions or prescriptions that play a similar role to the criterion of simplicity of Dirac. We illustrate this aspect by considering two of the more simple derivations, namely, the one of Barut [13–15] and the other of Landau and Lifshitz [7]. The idea in Barut's derivation is to add the electron's own field to the external electromagnetic field that appears in the Lorentz force. This procedure cannot be carried out directly because the self-field of the electron is singular when it is evaluated at the electron position. To deal with this trouble Barut evaluates the electron field, given by the Lienard–Wiechert formula, on the electron worldline and considers that the retarded point that appears in the Lienard–Wiechert formula is also on the electron worldline and very close to the position of the electron. Strictly speaking, this is an improper procedure since the second point is certainly not the retarded point associated with the first one. Nevertheless, the procedure is successful because in the limit when the two points coincide the Lorentz–Dirac equation is obtained, after hiding a divergent quantity through the renormalization of the electron mass. Thus, Barut's method cannot be justified starting from the first principles, and the special way by means of which the self-field of the electron is taken into account can be considered as replacing the criterion of simplicity of Dirac.

Landau and Lifshitz [7] proposed to incorporate the effect of the electron's own field by constructing a 4-vector orthogonal to the electron's 4-velocity and that reduces in the non-relativistic limit to the already known form of the radiation reaction force, obtaining in this way the Lorentz–Dirac equation. But this procedure, by no means, implies that that the 4-vector B_μ in equation (1.1) is equal to zero since B_μ of equation (1.5) as well as those presented in section 3 do not alter the non-relativistic limit of the second term on the right-hand side of equation (1.1).

As will be shown in the next section, the study of the 4-momentum associated with the electron field allows us to see in a clear way that the criterion of simplicity corresponds to a special choice or prescription to deal with the singular nature of the electron field. The existence of arbitrary dimensionless parameters in the 4-vector B_μ may appear to be disappointing, but they are an unavoidable consequence of the point model for the electron. In particular, in the Eliezer equation, that is, equation (1.1) with B_μ in equation (1.5), each different choice of the parameter k_2 gives rise to a different trajectory in the same external field. Nevertheless, this situation does not have catastrophic consequences because, as will be shown in section 5, for reasonable choices of the dimensionless parameters the different trajectories cannot be discriminated from the trajectory determined by the Lorentz–Dirac equation in a relevant practical situation as in a synchrotron.

2. The electron 4-momentum

In order to obtain physical insight into the origin of the 4-vector B_μ in equation (1.1), it is convenient to derive this equation starting from the 4-momentum associated with the retarded Lienard–Wiechert field generated by a point electron. In what follows we outline such a derivation. As usual in a relativistic field theory, the 4-momentum P_μ is given by

$$P_\mu = \frac{1}{c}\int_\Sigma T_{\mu\nu}\,\mathrm{d}\Sigma^\nu, \tag{2.1}$$

where $T_{\mu\nu}$ is the energy–momentum tensor determined by the electron's retarded Lienard–Wiechert electromagnetic field, and Σ is any space-like surface that intercepts the electron worldline at the point $z_\mu(\tau)$. The energy–momentum tensor $T_{\mu\nu}$, like the retarded Lienard–Wiechert field, is singular when the field point x^μ is located on the electron worldline. In particular then, strictly speaking, P^μ of (2.1) is meaningless because the integrand is singular at the point $z_\mu(\tau)$ where the space-like surface Σ cuts the electron worldline. Therefore, in order to give a meaning to (2.1) it is absolutely necessary to isolate the singular point $z_\mu(\tau)$ by means of a two-dimensional surface σ contained in Σ and that encloses this point, followed by a limiting procedure by means of which σ is shrunk to $z_\mu(\tau)$.

A deeper insight on the physical meaning of the different terms that appear in the enlarged Dirac's equation of motion (1.1) is obtained with the help of Teitelboim's splitting of the energy–momentum tensor [16, 17], namely,

$$T_{\mu\nu} = T^r_{\mu\nu} + T^b_{\mu\nu}, \tag{2.2}$$

where $T^r_{\mu\nu}$ contains the terms of $T_{\mu\nu}$ that behave as ρ^{-2} in the invariant distance $\rho = v_\mu(x^\mu - z^\mu)/c$, while $T^b_{\mu\nu}$ is built up with the terms that behave like ρ^{-3} and ρ^{-4}. Both parts of $T_{\mu\nu}$ satisfy

$$\partial^\nu T^r_{\mu\nu} = \partial^\nu T^b_{\mu\nu} = 0, \tag{2.3}$$

off the electron worldline. This means that there is no interchange of energy–momentum between $T^r_{\mu\nu}$ and $T^b_{\mu\nu}$ in any four-dimensional volume that excludes the electron worldline. Thus, $T^r_{\mu\nu}$ and $T^b_{\mu\nu}$ have dynamical independence everywhere except perhaps on the electron worldline. In addition, $T^r_{\mu\nu}$ has no flux across light cones emanating from the electron worldline into the future. The above properties of $T^r_{\mu\nu}$ imply that it represents energy–momentum that detaches itself from the electron and leads an independent existence as soon as it is produced by the electron [16, 17]. In other words, $T^r_{\mu\nu}$ describes the radiation emitted by the electron.

The splitting (2.2) induces a natural splitting of the 4-momentum (2.1) into the form

$$P_\mu = P^r_\mu + P^b_\mu. \tag{2.4}$$

The evaluation of P^r_μ leads to

$$P^r_\mu = (2e^2/3c^5)\int_{-\infty}^{\tau} \dot{v}^2 v_\mu\,\mathrm{d}\tau. \tag{2.5}$$

It turns out that P^r_μ is free of any ambiguity whatsoever since it is independent of the form of the space-like surface Σ as well as of the two-dimensional surface σ and the limiting procedure by means of which σ is shrunk to the point $z_\mu(\tau)$. All that matters is the point $z_\mu(\tau)$ where Σ intercepts the electron worldline.

In contradistinction with $T^r_{\mu\nu}$, the part $T^b_{\mu\nu}$ of $T_{\mu\nu}$ is such that it has a non-vanishing flux across the light cones emanating from the electron worldline into the future. Besides, $T^b_{\mu\nu}$ satisfies the notable identity [18]

$$T^b_{\mu\nu} = \partial^\lambda K_{\mu\nu\lambda}, \tag{2.6}$$

where $K_{\mu\nu\lambda}$ is antisymmetric in indices ν and λ and depends only on retarded quantities. Because of the Stokes theorem, equation (2.6) permits us to transform the integral over the space-like surface Σ that appears in the definition of P^b_μ into an integral over the two-dimensional surface σ plus an integral over a closed two-dimensional surface that lies on Σ and is located at spacial infinity. It is easy to see that this last integral vanishes for an electron with a uniform motion in the remote past. Thus, only the integral over the two-dimensional surface σ remains. In particular, this means that P^b_μ, in contradistinction with P^r_μ of equation (2.5), depends only on the proper time τ at which Σ cuts the electron worldline. The physical interpretation of $T^b_{\mu\nu}$ is then obvious; it represents energy momentum that is tied to the electron and is carried along with it. Consequently, P^b_μ can be identified with the electron 4-momentum. Unfortunately, however, due to the strong singularities of $T^b_{\mu\nu}$, the bound 4-momentum P^b_μ is, in contradistinction with P^r_μ, highly dependent on the form of Σ near $z_\mu(\tau)$ as well as on the choice of the two-dimensional surface σ and the limiting procedure by means of which σ is shrunk to the point $z_\mu(\tau)$. Precisely, these peculiarities are those that appear reflected in the existence of a whole family of 4-vectors B_μ in equation (1.1).

If in a vicinity of $z_\mu(\tau)$, the space-like surface Σ and the two-dimensional surface σ are chosen as in Dirac's paper, that is, if in a vicinity of $z_\mu(\tau)\Sigma$ coincides with the plane orthogonal to the 4-velocity v_μ and σ is the sphere of radius ϵ centred at the electron position at the proper time τ, then the following value for P^b_μ is obtained [16, 17]:

$$P^b_\mu = mv^\mu - (2e^2/3c^3)\dot{v}_\mu, \tag{2.7}$$

where m is the renormalized mass, that is, $m = m_0 + (e^2/2\epsilon c^2)$, with m_0 the bare mass of the electron. From equations (2.5) and (2.7), the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{P^r_\mu(\tau) + P^b_\mu(\tau)\right\} = (e/c)F_{\mu\nu}v^\nu \tag{2.8}$$

becomes the Lorentz–Dirac equation. In other words, the above choice of Σ and σ leads to a 4-vector $B_\mu = 0$ in equation (1.1). But this choice is not imperative from any fundamental physical principle; the only critical restriction on the space-like surface Σ is that it must cut the electron worldline orthogonally in order to make the electron mass renormalization possible [19]. Other choices for Σ and σ will lead, in general, to a 4-vector B_μ different from zero. We will not derive B_μ of equation (1.5) in this way, but it is not difficult to visualize the origin of the dimensionless parameter k_2 in equation (1.5). For example, it is possible to choose as the two-dimensional surface σ an ellipsoid of revolution around the 4-acceleration $\dot{v}_\mu$ in the plane orthogonal to the 4-velocity v_μ. Now, if the lengths of the axes of this ellipsoid are chosen as $a = \epsilon$ and $b = k\epsilon$, then the ellipsoid will coincide with the 2-sphere of Dirac when the dimensionless parameter k is equal to one. But in the general case, when the ellipsoid is shrunk to the point $z_\mu(\tau)$, the limiting value will be a function of k because of the strong singularities of $T^b_{\mu\nu}$. Therefore. the dimensionless parameters that appear in the 4-vector B_μ are related not only to the singular nature of the electron fields, but also to geometrical ingredients contained in the surfaces Σ and σ and the way in which σ is shrunk to the point $z_\mu(\tau)$ [20].

3. Enlarged Lorentz–Dirac equations

The construction of 4-vectors B_μ corresponding to powers of τ_0 higher than 2 does not present difficulties. In fact, B_μ in (1.1) has dimensions of 4-velocity, and therefore a balance exists in (1.5) between the number of dots in each term inside the brackets and the power of τ_0 in front of it. Thus, the number of dots over $\dot{v}^2 = \dot{v}_\alpha\dot{v}^\alpha$ is two, as is the number of dots in the term $\ddot{v}_\mu$

and the power of τ_0 in front of the bracket. For the construction of B_μ for higher powers of τ_0 it is convenient to consider that the 4-velocity v_μ satisfies the following relations:

$$\begin{aligned} &v^2 = -c^2, \\ &(v\dot{v}) = 0, \\ &(v\ddot{v}) = -\dot{v}^2, \\ &(v\dddot{v}) = -3(\dot{v}\ddot{v}), \\ &(vv^{(4)}) = -3\ddot{v}^2 - 4(\dot{v}\,\dddot{v}), \\ &(vv^{(5)}) = -10(\ddot{v}\,\dddot{v}) - 5(\dot{v}v^{(4)}), \end{aligned} \tag{3.1}$$

where the notation $v^{(4)}$, $v^{(5)}$ is used for the derivatives of the 4-velocity of orders 4 and 5 respectively, and besides $v^2 = v_\alpha v^\alpha$, $(v\dot{v}) = v_\alpha \dot{v}^\alpha$, $(v\ddot{v}) = v_\alpha \ddot{v}^\alpha$, etc. The tentative form for $B_\mu(3)$ proportional to τ_0^3 is

$$\tau_0^3\{a(\dot{v}\ddot{v})v_\mu/c^2 + b\dot{v}^2\dot{v}_\mu/c^2 + \mathrm{d}\,\dddot{v}_\mu\}, \tag{3.2}$$

where a, b and d are dimensionless numbers and c is the speed of light. Equation (1.3) imposes the following restriction on a, b and d:

$$b\dot{v}^4/c^2 + (a+3d)\ddot{v}^2 + (a+4d)(\dot{v}\,\dddot{v}) = 0. \tag{3.3}$$

For an arbitrary motion and any proper time, this equation has the unique solution $a = b = d = 0$. This means that no 4-vector B_μ proportional to τ_0^3 exists. The situation is different for B_μ proportional to τ_0^4. A tentative $B_\mu(4)$ is the following:

$$B_\mu(4) = \tau_0^4\{a_1\dot{v}^4 v_\mu/c^4 + b_1(\dot{v}\ddot{v})\dot{v}_\mu/c^2 + d_1\dot{v}^2\ddot{v}_\mu/c^2\}, \tag{3.4}$$

where a_1, b_1 and d_1 are dimensionless numbers. The imposition of (1.3) leads to $d_1 = -(4a_1 + b_1)/5$, and consistency with the conservation of angular momentum finally gives

$$B_\mu(4) = k_4\tau_0^4\{7\dot{v}^4 v_\mu/c^4 - 8(\dot{v}\ddot{v})\dot{v}_\mu/c^2 - 4\dot{v}^2\ddot{v}_\mu/c^2\}, \tag{3.5}$$

where k_4 is an arbitrary dimensionless parameter, in general unrelated to k_2. In fact, it is easy to see that (3.5) satisfies equations (1.3) and (1.4), with

$$A_{\mu\lambda}(4) = -4k_4\tau_0^4\{\dot{v}^2(v_\mu\dot{v}_\lambda - v_\lambda\dot{v}_\mu)/c^2\}. \tag{3.6}$$

However, (3.5) is not the unique admissible B_μ of order τ_0^4; another tentative form is the following:

$$\bar{B}_\mu(4) = \tau_0^4\left\{a_2\ddot{v}^2 v_\mu/c^2 + b_2(\dot{v}\,\dddot{v})v_\mu/c^2 + d_2 v_\mu^{(4)}\right\}, \tag{3.7}$$

where a_2, b_2 and d_2 are dimensionless numbers and c is, as usual, the speed of light. The imposition of (1.3) leads to $b_2 = 2a_2$ and $d_2 = -2a_2/5$. This reduces equation (3.7) to

$$\bar{B}_\mu(4) = \bar{k}_4\tau_0^4\left\{5\ddot{v}^2 v_\mu/c^2 + 10(\dot{v}\,\dddot{v})v_\mu/c^2 - 2v_\mu^{(4)}\right\}, \tag{3.8}$$

where $\bar{k}_4$ is an arbitrary dimensionless parameter, unrelated to k_4. It is easily seen that $\bar{B}_\mu(4)$ of equation (3.8) also fulfils the restriction (1.4) with $\bar{A}_{\mu\lambda}(4)$ given by

$$\bar{A}_{\mu\lambda}(4) = -2\bar{k}_4\tau_0^4\{v_\mu\dddot{v}_\lambda - v_\lambda\dddot{v}_\mu + \ddot{v}_\mu\dot{v}_\lambda - \ddot{v}_\lambda\dot{v}_\mu\}. \tag{3.9}$$

It is rather clear that, in general, there exist 4-vectors B_μ of arbitrary powers of τ_0 since conditions (1.3) and (1.4) are not very restrictive. However, in order to understand the essential features of the enlarged Lorentz–Dirac equation (1.1), it is enough to study in detail the equations associated with B_μ of (1.5), (3.5) and (3.8).

4. Exact solutions

In this section, we will show that the enlarged Lorentz–Dirac equations (1.1) with B_μ given in (1.5), (3.5) and (3.8) have exact solutions for the circular motion described in section 1. The coordinates axes are chosen in such a way that the circular orbit of radius a lies in the x–y plane and is centred at the origin, and the z-axis is perpendicular to the plane of the orbit. The non-vanishing components of the external electromagnetic field are therefore

$$F^{01} = E_x, \qquad F^{02} = E_y, \qquad F^{12} = B_z. \tag{4.1}$$

The solutions that we are looking for describe the motion of an arbitrary number N of identical charges that are rotating at a constant angular velocity ω. The charges are equally spaced on a circumference of radius a. Because of the symmetry of the motion, the tangential and radial components of the force acting on a charge are the same for any charge; therefore, it is enough to consider the motion of only one of them, which we choose to be on the x-axis at $t = 0$:

$$x^0 = ct, \qquad x^1 = a\cos\omega t, \qquad x^2 = a\sin\omega t, \qquad x^3 = 0. \tag{4.2}$$

Besides the external field acting on the charge whose tentative trajectory is defined by (4.2), we have to consider the retarded fields of the rest of the charges. To this end, we will label the charge of (4.2) with the number N, while we will label the rest of the charges by the natural number k, with $k = 1, 2, 3, \ldots, N-1$, increasing its value in the counter-clockwise direction, that is, in the direction of the motion, since we are considering the charge $e > 0$. At time t, the radial direction associated with the charge number k makes an angle $(2\pi/N)k$ with the radial direction associated with the charge number N. But the field strengths of the charge k that are acting on the charge N are determined by the retarded position of the charge k. If we denote by $2\alpha_k$ the angle that the radial direction associated with the retarded position of charge k makes with the radial direction associated with the charge number N, then $2\alpha_k < (2\pi/N)k$ and the retardation condition, which states that the time that the charge k takes to travel from its retarded position to its actual position at time t is the same as the time that the light takes to travel from the retarded position of charge k to the actual position of the charge N, translates into the following relation:

$$k\pi/N = \alpha_k + \beta\sin\alpha_k, \tag{4.3}$$

where $\beta = a\omega/c$. Equation (4.3) determines the angle α_k in a unique way in terms of the discrete variable k and the continuous variable β [3]. Of course, α_k is time independent. If we denote by E_{kx}, E_{ky}, E_{kz}, B_{kx}, B_{ky} and B_{kz} the Cartesian components of the retarded field due to the charge k, we obtain, starting from the Lienard–Wiechert field strengths of a point charge, the following expressions for the components that are different from zero:

$$\begin{aligned}
E_{kx} &= f_k\{\sin(\omega t + \alpha_k) + \beta\sin(\omega t + 2\alpha_k)\} + g_k\cos(\omega t + \alpha_k),\\
E_{ky} &= -f_k\{\cos(\omega t + \alpha_k) + \beta\cos(\omega t + 2\alpha_k)\} + g_k\sin(\omega t + \alpha_k),\\
B_{kz} &= \left(\frac{e}{4a^2}\right)\frac{\beta\left\{s_k^2 + \beta^2\sin^2\alpha_k\right\}}{s_k^3\sin\alpha_k},
\end{aligned} \tag{4.4}$$

where the time-independent parameters f_k and g_k are given by

$$\begin{aligned}
f_k &= e/\gamma^2 R_k^2 s_k^3,\\
g_k &= e\beta^2(\beta + \cos\alpha_k)/aR_k s_k^3,
\end{aligned} \tag{4.5}$$

with

$$\begin{aligned}
\gamma &= (1-\beta^2)^{-1/2},\\
s_k &= 1 + \beta\cos\alpha_k,\\
R_k &= 2a\sin\alpha_k.
\end{aligned} \tag{4.6}$$

Therefore, the retarded fields of the charges are also of form (4.1).

In what follows, E_x and E_y are the Cartesian components of the total superposition of electric fields; similarly, B_z denotes the total magnetic field. Using (4.1) and (4.2) and B_μ given in (1.5), we obtain the following result for the $\mu = 1$ component of the enlarged Lorentz–Dirac equation (1.1):

$$-\cos\omega t = (ea/mc^2\beta^2\gamma)\{E_x + \beta B_z \cos\omega t\} + \lambda\beta\gamma(\beta^2\gamma^2 + 1)\sin\omega t - k_2\lambda^2\beta^2\gamma^2(\beta^2\gamma^2 + 2/3)\cos\omega t, \tag{4.7}$$

whereas the $\mu = 2$ component is

$$-\sin\omega t = (ea/mc^2\beta^2\gamma)\{E_y + \beta B_z \sin\omega t\} - \lambda\beta\gamma(\beta^2\gamma^2 + 1)\cos\omega t - k_2\lambda^2\beta^2\gamma^2(\beta^2\gamma^2 + 2/3)\sin\omega t, \tag{4.8}$$

where λ is the dimensionless parameter

$$\lambda = \frac{2e^2}{3mc^2a}. \tag{4.9}$$

The component $\mu = 0$ of (1.1) can be obtained starting from (4.7) and (4.8), and therefore can be ignored. Besides, the component $\mu = 3$ of (1.1) does not impose any restriction on the different parameters that characterize the motion under consideration.

If E denotes the magnitude of the external tangential electric field, the Cartesian components $E_x = -E\sin\omega t$ and $E_y = E\cos\omega t$ are, like the retarded components (4.4), time dependent. It is therefore more convenient to work in terms of the radial and tangential components E_ρ and E_ϕ of the total electric field, defined by

$$E_\rho = E_x\cos\omega t + E_y\sin\omega t, \qquad E_\phi = -E_x\sin\omega t + E_y\cos\omega t. \tag{4.10}$$

E_ρ and E_ϕ, in contradistinction to E_x and E_y, are time independent. In terms of them (4.7) and (4.8) become

$$E_\phi = (2e/3a^2)\beta^3\gamma^4 \tag{4.11}$$

$$E_\rho + \beta B_z = -(mc^2\beta^2\gamma/ae)\{1 - k_2\lambda^2\beta^2\gamma^2(\beta^2\gamma^2 + 2/3)\}. \tag{4.12}$$

Now, if $E_\phi(N)$, $E_\rho(N)$ and $B_z(N)$ are the components of the electromagnetic field due to the retarded fields of the remaining $N - 1$ charges, that is,

$$E_\rho(N) = \sum_{k=1}^{N-1} E_{k\rho}, \tag{4.13}$$

$$E_\phi(N) = \sum_{k=1}^{N-1} E_{k\phi}, \tag{4.14}$$

$$B_z(N) = \sum_{k=1}^{N-1} B_{kz}, \tag{4.15}$$

then equations (4.11) and (4.12) determine, in a unique way, the following values of the external electric and magnetic fields E and B respectively for which (4.2) constitutes an exact solution of the enlarged Lorentz–Dirac equation (1.1) with $B_\mu(2)$ of equation: (1.5)

$$E = (2e/3a^2)\beta^3\gamma^4 - E_\phi(N), \tag{4.16}$$

$$B = -(mc^2\beta\gamma/ae)\{1 - k_2\lambda^2\beta^2\gamma^2(\beta^2\gamma^2 + 2/3)\} - \beta^{-1}E_\rho(N) - B_z(N). \tag{4.17}$$

Following the same procedure, it is easy to show that (4.2) is also an exact solution of the enlarged Lorentz–Dirac equation (1.1) for B_μ given in (3.5). In this case, the values of the external fields that sustain the motion are

$$E = (2e/3a^2)\beta^3\gamma^4 - E_\phi(N) \tag{4.18}$$

$$B = -(mc^2\beta\gamma/ae)\{1 - k_4\lambda^4\beta^6\gamma^6(4 + 7\beta^2\gamma^2)\} - \beta^{-1}E_\rho(N) - B_z(N). \tag{4.19}$$

For the enlarged Lorentz–Dirac equation (1.1) with $\bar{B}_\mu(4)$ of (3.8), the values of the external fields that allow trajectory (4.2) as an exact solution are

$$E = (2e/3a^2)\beta^3\gamma^4 - E_\phi(N) \tag{4.20}$$

$$B = -(mc^2\beta\gamma/ae)\{1 + \bar{k}_4\lambda^4\beta^4\gamma^4(2 + 5\beta^2\gamma^2)\} - \beta^{-1}E_\rho(N) - B_z(N). \tag{4.21}$$

5. Discussion and comments

The enlarged Lorentz–Dirac equations (1.1) with B_μ of equation (1.5), (1.1) with B_μ of equation (3.5) and (1.1) with B_μ of equation (3.8) reduce to the Lorentz–Dirac equation if $k_2 = k_4 = \bar{k}_4 = 0$; therefore, the solutions (4.16), (4.17); (4.18), (4.19) and (4.20), (4.21) are also the exact solutions of the Lorentz–Dirac equation in this case. However, in the general case the solution of the Lorentz–Dirac equation for a given motion is different from the corresponding solution of the enlarged equations; nevertheless, the external electric field that sustains the motions is exactly the same, as is shown in equations (4.16), (4.18) and (4.20). This result is expected since the external electric field is directly related to the rate of radiation emitted by the system of charges. In fact, the external electric field $\mathbf{E}$ supplies the power $Ne\mathbf{v}\cdot\mathbf{E}$ to the charges, where $\mathbf{v}$ is the ordinary velocity. Besides, the kinetic energy of the charges remains constant and, due to the symmetry of the motion, the energy stored in the system of charges is also constant. Thus, the power that the external electric field supplies to the charges must be necessarily radiated away, and therefore $Ne\mathbf{v}\cdot\mathbf{E}$ is exactly the rate of radiation associated with the motion under consideration. This is also the rate of radiation obtained by computing the flux of the Poynting vector associated with the system of charges across a sphere of an infinitely large radius that encloses the orbiting charges [3]. The external electric fields given in equations (4.16), (4.18) and (4.20) are not only the same, but even more, they are completely independent of the dimensionless parameters that appear in the 4-vector B_μ of equations (1.5), (3.5) and (3.8). This result is fully consistent with the discussion of section 2, according to which the 4-vector B_μ arises from the energy–momentum that remains tied to the electron, which in turn is dynamically independent of the part of the energy–momentum that describes the radiation. The fact that the rate of radiation associated with an enlarged Lorentz–Dirac equation is the same as the one that follows from the Lorentz–Dirac equation means, in particular, that in the case of only one charge in a circular orbit the rate of radiation of any enlarged equation is $2e^2c\beta^4\gamma^4/3a^2$ and that when the number of charges N goes to infinity the rate of radiation tends to zero [3].

From equations (4.17), (4.19) and (4.21) it follows that the value of the external magnetic field, in contradistinction with the value of the external electric field, depends for a given motion on the choice of B_μ. This means that the trajectories of the different enlarged Lorentz–Dirac equations differ, in general, from each other and from the trajectory of the Lorentz–Dirac equation in the same external fields. In order to clarify this aspect, let us compare the magnetic field associated with the Lorentz–Dirac equation and the magnetic field (4.17) in the case of only one charge, that is,

$$B = -(mc^2\beta\gamma/ae)\{1 - k_2\lambda^2\beta^2\gamma^2(\beta^2\gamma^2 + 2/3)\}. \tag{5.1}$$

This magnetic field coincides with that of the Lorentz–Dirac equation if and only if $k_2 = 0$. Now, if we evaluate (5.1) for a given set of parameters e, m, a and ω with $k_2 \neq 0$, then this value of B and the value $E = (2e/3a^2)\beta^3\gamma^4$ determine an exact solution of the enlarged equation (1.1) with B_μ of equation (1.5) for one charge in a circular orbit of radius a. If we now use these values of E and B as external fields in the Lorentz–Dirac equation, the corresponding solutions of this equation will not, in general, describe a circumference of radius a, but, as will be shown below, the trajectory will be very close to it.

The motion of ultrarelativistic electrons in synchrotrons and storage rings constitutes the most prominent application of classical electrodynamics to a situation where the radiation emitted by the electron plays a central role. In these machines, the external electric and magnetic fields are not equal to the idealized external fields that define the exact solutions of this paper; nevertheless, they are of the same order of magnitude. So, without the need to perform a detailed study of the trajectory, formula (5.1) is useful in order to make an estimation of the change in the magnetic field implied by the enlarged equation (1.1) with B_μ of equation (1.5), with respect to the magnetic field prescribed by the Lorentz–Dirac equation in synchrotrons and storage rings. In order to illustrate this point, let us use the parameters of Cornell's electron synchrotron, namely $a = 10^4$ cm and $\gamma = 2 \times 10^4$, and let us take k_2 of the order of one. Then, the corresponding exact solution of the Lorentz–Dirac equation needs a magnetic field of 3333 G, and the difference with the magnetic field required by the same solution in the case of the enlarged equation (1.1) with B_μ of equation (1.5) is of the order of 10^{-13} G. Therefore, from a practical point of view both magnetic fields are the same since it is impossible at all to measure and control a magnetic field as small as 10^{-13} G. In other words, even if for given values of E and B the trajectories of the Lorentz–Dirac and the enlarged equation (1.1) with B_μ of equation (1.5) are not exactly the same, they cannot be discriminated in the relevant situation of synchrotrons and storage rings. If along the above lines we compare the exact solution of the Lorentz–Dirac equation with the corresponding solution of the enlarged equation (1.1) with B_μ of equation (3.5) for the same motion, the difference between the magnetic fields is now of the order of 10^{-30} G. Besides, it is rather evident that this difference becomes smaller if we we consider enlarged equations with B_μ of higher powers of τ_0, since τ_0 is a very small number, of the order of 10^{-23} s. Therefore, since the effect of B_μ is so small, it can be inferred that in synchrotrons and storage rings, the use of an enlarged Lorentz–Dirac equation would result, from a practical point of view, in the same trajectories that follow from the Lorentz–Dirac equation.

The conservation of energy, momentum and angular momentum, the Maxwell equations and the point model for the electron allow the construction of expressions for B_μ that contain derivatives of arbitrarily high order in the 4-velocity. However, these fundamental principles do not allow us to privilege B_μ proportional to a particular power of τ_0 nor to determine the values of the dimensionless parameters that appear in it. Thus, instead of a unique equation of motion, classical electrodynamics gives rise to a huge number of them, for instance, by choosing in equation (1.5) any value for k_2 in a continuous range around the value $k_2 = 1$. This in turn gives rise to a bundle of different trajectories in the same external fields. However, we have showed that this situation does not present practical problems in the design or operation of synchrotrons and storage rings. Therefore, the preconception that classical electrodynamics must determine a unique and perfectly well-defined trajectory for an electron in a specific external electromagnetic field not only does not follow from the three essential components of Dirac's approach, but it is not even needed in experimental and practical applications of classical electrodynamics which, in definitive, are the supreme arbitrators in physics. Let us remark that the design and operation of synchrotrons is actually realized with equation (1.1) where not only the 4-vector B_μ is neglected, but also the term $\ddot{v}_\mu$ since this last term is much smaller than the term $\dot{v}^2 v_\mu/c^2$ for ultrarelativistic electrons [21, 22], the case in which equation (1.1) becomes Bonnor's equation [8].

The self-accelerating solutions of an enlarged equation are eliminated, like in the case of the Lorentz–Dirac equation, by imposing the condition of inertial motion in the remote future. To make this point clear, it is useful to consider the motion of a charge along a straight line. Let us then consider a time-independent electric field whose only component $E(x)$ points along the positive x-axis, and in addition vanishes identically outside the interval $0 \leqslant x \leqslant l$. For definiteness, let us choose a charge $e > 0$ and consider the Eliezer equation, that is, equation (1.1) with B_μ of equation (1.5). In this case, it is convenient to write the non-vanishing components of the 4-velocity in terms of the rapidity $w(\tau)$ as follows [23]:

$$v^0 = c\cosh(w/c), \qquad v^1 = c\sinh(w/c). \tag{5.2}$$

Then, since $F^{01} = E$ is the only non-vanishing component of the field strengths $F^{\mu\nu}$, equation (1.1) becomes

$$\dot{w} - \tau_0 \ddot{w} = f(\tau) + k_2\tau_0^2(\dot{w}^3/3c^2 - 2\,\dddot{w}/3), \tag{5.3}$$

where $f(\tau) = eE/m$. In the case of the Lorentz–Dirac equation, that is, for $k_2 = 0$, the solution of equation (5.3) with $\dot{w}(\tau) = 0$ in the remote future is

$$\dot{w} = \frac{e^{\tau/\tau_0}}{\tau_0}\int_\tau^\infty e^{-\tau'/\tau_0} f(\tau')\,d\tau'. \tag{5.4}$$

In fact, if τ_1 is the value of the proper time at which the charge reaches the point $x = l$, then (5.4) implies that $\dot{w} = 0$ for any $\tau > \tau_1$. In the case of $k_2 \neq 0$, the solution of (5.3) satisfies the following integro-differential equation:

$$\dot{w} = \frac{e^{\tau/\tau_0}}{\tau_0}\int_\tau^\infty e^{-\tau'/\tau_0}\left\{f(\tau') + k_2\tau_0^2(\dot{w}^3/3c^2 - 2\,\ddot{w}/3)\right\}d\tau', \tag{5.5}$$

whose solution can be found by successive iterations starting from the solution $w_1(\tau)$ of (5.4). If $w_1(\tau)$ is introduced in the integrand of (5.5), the first iteration leads to a solution $w_2(\tau)$ which satisfies $\dot{w}_2(\tau) = 0$ for $\tau > \tau_1$. Now, since this property is valid for any iteration, the solution of (5.3) will be such that $\dot{w}(\tau)$ vanishes in the remote future. If $\tau = 0$ is chosen as the proper time corresponding to the point $x = 0$, then from equation (5.5) it follows that, in general, $\dot{w}(\tau)$ is different from zero for $\tau < 0$, in spite of that for $x < 0$ there is no electric field. The amount of acausality is a function of the 4-vector B_μ under consideration. However, the idea [24] that preacceleration can be eliminated in an enlarged equation that considers all the permissible B_μ in the form of a power series in the parameter τ_0 does not seem feasible because of the highly nonlinear character of each B_μ and the infinitely large number of arbitrary dimensionless parameters that would be present in such a case. Thus, the violation of causality appears to be an innate feature of the point model for a charge in classical electrodynamics.

From the point of view that the quantum theory of radiation is more fundamental than the corresponding classical theory, it may seem that to study the dynamics of a point electron in classical electrodynamics is unfruitful. However, the simpler mathematical formalism of the classical theory, together with its more direct physical interpretation, may shed light into related problems of the quantum theory. The fact that the three fundamental components of the

Dirac classical theory of the electron also play a central role in the quantum theory of radiation constitutes a clear indication that both theories are profoundly connected. Unfortunately, however, the intricate mathematical formalism of quantum electrodynamics has made difficult to make this connection explicit. So, it is not at all meaningless to attempt to throw light into this question by starting from the classical theory of radiation. We will illustrate this aspect by showing that classical electrodynamics suggests, in a rather natural way, a new approach to deal with tiny effects as the Lamb shift and the anomalous magnetic moment of the electron in the spectrum of the hydrogen atom. To this end, we will focus our attention on the most simple, direct and natural choice for the 4-momentum of the electron, that is, the 4-momentum associated with the Lorentz–Dirac equation given in equation (2.7). This formula shows that the 4-momentum of the electron is mv_μ only if the electron is in a uniform motion, that is, for the case of a free electron. However, when the electron is under the influence of an external field its 4-acceleration $\dot{v}_\mu$ is different from zero and the 4-momentum of the electron has an additional contribution given by the second term on the right-hand side of equation (2.7). In order to make an estimate of the importance of the additional contribution with respect to the main one given by mv_μ, let us consider the case of a synchrotron, which is relevant in classical electrodynamics. For definiteness, let us choose the parameters of the Cornell synchrotron, which has a radius $r = 10^4$ cm and a factor $\gamma = 2 \times 10^4$. The spatial part of the second term in (2.7) is of the order of $2e^2\gamma^2 a/3c^3$, where the acceleration a is of the order of 10^{17} cm s^{-2}. Therefore, the second term of (2.7) is of the order of 10^{-25} g cm s^{-1}, while the main contribution is of the order of $mc\gamma \sim 10^{-13}$ g cm s^{-1}. This means that the contribution to the electron 4-momentum due to the energy–momentum tied to the electron is 12 orders of magnitude smaller than the main part $mc\gamma$, and therefore it is completely negligible. In particular, this explains why the term proportional to $\ddot{v}_\mu$ in equation (1.1) is neglected in the design of a synchrotron.

In section 2 it was clearly stated that from a physical point of view the second term in (2.7) has its sources (like the first term) in the energy–momentum that remains tied to the electron, so it is natural to expect that this picture is also valid in a quantum description. The hydrogen atom is, certainly, a quantum system, but in spite of this equation (2.7) is still useful to obtain the order of magnitude of the terms that appear in it by considering the average value $\bar{v}$ of the velocity in the fundamental state of the hydrogen atom. As is known, $\bar{v}$ is of the order of 3×10^8 cm s^{-1}. Besides, the hydrogen atom is weakly relativistic and for this reason equation (2.7) can be written as follows:

$$\mathbf{P}^b = m\mathbf{v} - (2e^2/3c^3)\mathbf{a}, \tag{5.6}$$

where $\mathbf{v}$ and $\mathbf{a}$ are the ordinary velocity and acceleration, respectively. Since the Bohr radius is of the order of 10^{-8} cm, the average acceleration $\bar{a}$ is of the order of 10^{25} cm s^{-2}. Thus, the second term on the right-hand side of equation (5.6) is of the order of 10^{-25} g cm s^{-1}, while the main part $m\bar{v}$ is of the order of 10^{-19} g cm s^{-1}. Therefore, the additional contribution to $m\bar{v}$ in (5.6) is only six orders of magnitude smaller than the main one. This result, together with the fact that the spectrum of the hydrogen atom is measured with very high accuracy, means that, in contradistinction with the case of a synchrotron, in the case of the hydrogen atom the contribution of the second term in (5.6) cannot be neglected. This aspect can be clarified by comparing the additional momentum given by the second term in (5.6) with the momentum $e\mathbf{A}/c$ associated with a uniform magnetic field H along the z-axis, that is, with $A \sim rH$, where the radius r is of the order of the Bohr radius. Thus, in the hydrogen atom case, the additional momentum of 10^{-25} g cm s^{-1} is nearly the same momentum as the one associated with a homogeneous magnetic field of the order of 1000 G, which, as is known, gives rise to a splitting of the order of 10^{-5} eV in the spectrum of the hydrogen atom. But the splitting due to the Lamb shift is also, roughly speaking, of this order of magnitude, and therefore it seems rather natural to expect that the additional momentum in (5.6) is related to the Lamb shift. This expectation is reinforced by taking into account that in quantum electrodynamics the cloud of virtual photons that surrounds the electron plays a central role in the calculation of the Lamb shift and the anomalous magnetic moment. The exact evaluation of the Lamb shift requires, of course, a quantum formulation. According to the above view, the customary momentum operator of quantum mechanics $\mathbf{p} = -\mathrm{i}\hbar\nabla$ is exact only for a free electron, and in the case of the hydrogen atom it should be replaced by the quantum version of (5.6). In this conception, it seems rather natural to calculate the Lamb shift as well as the anomalous magnetic moment starting from the Pauli equation, with the corresponding relativistic corrections, and where $\mathbf{p} = -\mathrm{i}\hbar\nabla$ is replaced by $\mathbf{p} = -\mathrm{i}\hbar\nabla + \mathbf{q}$, where $\mathbf{q}$ is the quantum operator, obtained with the help of the techniques of quantum mechanics, associated with the second term on the right-hand side of equation (5.6). This work is in progress and the results will be presented in a forthcoming paper.

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